

# SOBOLEV INEQUALITY ON RIEMANNIAN MANIFOLDS\*\*

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## Abstract

Let  $M$  be an  $n$  dimensional complete Riemannian manifold satisfying the doubling volume property and an on-diagonal heat kernel estimate. The necessary-sufficient condition for the Sobolev inequality  $\|f\|_q \leq C_{n,\nu,p,q}(\|\nabla f\|_p + \|f\|_p)$  ( $2 \leq p < q < \infty$ ) is given.

**Keywords** Sobolev inequality, Complete manifold, Riesz transform, Potential, Heat kernel

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## § 1. Introduction

Let  $M$  be an  $n$  dimensional complete Riemannian manifold,  $\rho$  be the geodesic distance on  $M$ , and  $d\mu$  be Riemannian measure. Denote by  $B(x, r)$  the geodesic ball of center  $x \in M$  and radius  $r > 0$ , and by  $V(x, r)$  its Riemannian volume.

One says that  $M$  satisfies the doubling volume property if there exists a constant  $D_0$  such that

$$V(x, 2r) \leq D_0 V(x, r), \quad \forall x \in M, r > 0. \quad (1.1)$$

We denote  $\nu = \log_2 D_0$ .

Let  $\Delta$  be the Laplace-Beltrami operator on  $M$ ,  $H(x, y, t)$  be the heat kernel on  $M$ .

Using the boundedness of the potential  $(I + (-\Delta)^{\frac{1}{2}})^{-1}$  and the Riesz transform, Li [1] obtained that for  $1 < p < q < \infty$ , the Sobolev inequality

$$\|f\|_q \leq C_{n,p,q}(\|\nabla f\|_p + \|f\|_p), \quad \forall f \in H_1^p(M) \quad (1.2)$$

holds on a complete manifold with non-negative Ricci curvature if and only if

$$\begin{cases} \inf_{x \in M} V(x, 1) > 0, \\ \frac{1}{p} - \frac{1}{n} \leq \frac{1}{q} < \frac{1}{p}. \end{cases}$$

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In this note, we replace the condition  $\text{Ricci } M \geq 0$  by the doubling volume property and an upper bounds for the heat kernel. We obtain the following result:

**Theorem 1.1.** *Let  $M$  be an  $n$  dimensional complete Riemannian manifold satisfying the doubling volume property (1.1) and the heat kernel  $H(x, y, t)$  at  $M$  satisfies*

$$H(x, x, t) \leq \frac{C}{V(x, \sqrt{t})}, \quad \forall x \in M, t > 0 \quad (1.3)$$

for some  $C > 0$ . Then for  $2 \leq p < q < \infty$ , the Sobolev inequality

$$\|f\|_q \leq C_{n,\nu,p,q}(\|\nabla f\|_p + \|f\|_p), \quad \forall f \in H_1^p(M) \quad (1.4)$$

holds for some constant  $C_{n,\nu,p,q} > 0$  if and only if

$$\begin{cases} \inf_{x \in M} V(x, 1) > 0, \\ \frac{1}{p} - \frac{1}{n} \leq \frac{1}{q} < \frac{1}{p}. \end{cases}$$

Our assumption on  $M$ , apart from the doubling volume property, is the heat kernel estimate (1.3). From this on-diagonal estimate, the corresponding off-diagonal estimate automatically follows (see [5, Theorem 1.1]): for any  $\alpha \in (0, \frac{1}{4})$ ,

$$H(x, y, t) \leq \frac{C_\alpha}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}} e^{-\alpha \frac{\rho^2(x,y)}{t}}, \quad \forall x, y \in M, t > 0 \quad (1.5)$$

for some  $C_\alpha > 0$ .

With the doubling volume property, this implies that for any  $\alpha \in (0, \frac{1}{4})$ ,

$$H(x, y, t) \leq \frac{C'_\alpha}{V(x, \sqrt{t})} e^{-\alpha \frac{\rho^2(x,y)}{t}} \quad (1.6)$$

for some  $C'_\alpha > 0$ . Indeed  $B(y, \sqrt{t}) \subset B(x, \sqrt{t} + \rho(x, y))$ . Now an obvious consequence of the doubling volume property is

$$V(x, \theta r) \leq D_0 \theta^\nu V(x, r), \quad \forall \theta > 1. \quad (1.7)$$

Therefore

$$V(y, \sqrt{t}) \leq V(x, \sqrt{t} + \rho(x, y)) \leq c \left(1 + \frac{\rho(x, y)}{\sqrt{t}}\right)^\nu V(x, \sqrt{t}),$$

and the estimate follows.

In fact, we should have  $\nu \geq n$ . Since  $\lim_{r \rightarrow 0} \frac{V(x, r)}{r^n} = \Omega_n > 0$ , we have (see (1.7))

$$\frac{V(x, 1)r^\nu}{D_0} \leq V(x, r) \leq C_n r^n, \quad r \ll 1.$$

**Example 1.1.** The assumptions of Theorem 1.1 are satisfied on manifolds where a parabolic Harnack principle holds (see [6, Chapter 5]).

It is easy to construct manifolds that satisfy the assumptions of Theorem 1.1, but where the parabolic Harnack principle is false. A typical example is the following (see [3]): take two copies of  $\mathbb{R}^2 \setminus B(0, 1)$ , and glue them smoothly along the unit circles.

By [6, Theorem 5.6.4 and Theorem 5.6.5], we conclude that the parabolic Harnack principle is satisfied on manifolds with non-negative Ricci curvature. Thus Theorem 1.1 is an extension of the result in [1] when  $2 \leq p < +\infty$ .

§ 2. Proof of Theorem 1.1

We consider the potential  $(I + (-\Delta)^{\frac{1}{2}})^{-1}$ . For  $1 \leq p \leq \infty$  and  $\forall f \in L^p(M)$ , we have

$$Tf \triangleq (I + (-\Delta)^{\frac{1}{2}})^{-1}f(x) = \int_0^\infty e^{-t} \int_M P(x, y, t)f(y)dydt, \tag{2.1}$$

where  $P(x, y, t)$  is the Poisson kernel on  $M$ , i.e.,

$$P(x, y, t) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s} s^{-\frac{1}{2}} H\left(x, y, \frac{t^2}{4s}\right) ds = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x, y, s) ds.$$

**Proposition 2.1.** *There exists some constant  $C_p > 0$  such that*

$$\|Tf\|_{L^p(M)} \leq C_p \|f\|_{L^p(M)}, \quad \forall f \in L^p(M).$$

**Proof.** By Theorem 3.5 in [2],  $\int_M H(x, y, t) \leq 1$ , so  $\int_M P(x, y, t) \leq 1$ . Therefore  $T$  is  $L^p$  ( $1 \leq p \leq \infty$ ) bounded.

Next, we will show that  $T$  is also bounded from  $L^p(M)$  to  $L^q(M)$  for  $\frac{1}{p} - \frac{1}{n} = \frac{1}{q}$ . Let

$$k(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty te^{-t} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x, y, s) dsdt, \tag{2.2}$$

$$k_1(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty te^{-t} \int_0^1 e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x, y, s) dsdt, \tag{2.3}$$

$$k_2(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty te^{-t} \int_1^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x, y, s) dsdt, \tag{2.4}$$

$$T_1f(x) \triangleq (I + (-\Delta)^{\frac{1}{2}})^{-1}_1 f(x) = \int_M k_1(x, y)f(y)d\mu(y),$$

$$T_2f(x) \triangleq (I + (-\Delta)^{\frac{1}{2}})^{-1}_2 f(x) = \int_M k_2(x, y)f(y)d\mu(y).$$

Clearly

$$k(x, y) = k_1(x, y) + k_2(x, y), \tag{2.5}$$

$$Tf(x) = \int_M k(x, y)f(y)d\mu(y) = T_1f(x) + T_2(x). \tag{2.6}$$

First, we state the following two lemmas.

**Lemma 2.1.** *Let  $M$  be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Let  $\inf_{x \in M} V(x, 1) = \delta > 0$ ,  $1 < p < q < \infty$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ . Then  $T_1$  is of type  $(p, q)$ , i.e.  $T_1$  is bounded from  $L^p(M)$  to  $L^q(M)$ .*

**Lemma 2.2.** *Let  $M$  be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Let  $\inf_{x \in M} V(x, 1) = \delta > 0$ ,  $1 < p < q < \infty$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ . Then  $T_2$  is of type  $(p, q)$ .*

Next, once we established Lemmas 2.1–2.2, with the  $(p, p)$  boundedness of  $T$  (see Proposition 2.1) and Marcinkiewicz interpolation theorem (see [4] for example), we obtain

**Theorem 2.1.** *Let  $M$  be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Let  $1 < p, q < \infty$ , and  $\inf_{x \in M} V(x, 1) > 0$ ,  $\frac{1}{p} - \frac{1}{n} \leq \frac{1}{q} \leq \frac{1}{p}$ . Then  $T$  is of type  $(p, q)$ .*

Furthermore, we can get

**Theorem 2.2.** *Let  $M$  be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Then for  $2 \leq p < \infty$ , there exists a constant  $C_{n,p} > 0$ , such that*

$$\|(-\Delta)^{\frac{1}{2}} f\|_p \leq C_{n,p} \|\nabla f\|_p, \quad \forall f \in C_0^\infty(M).$$

**Proof.** Let  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then for all  $g \in C_0^\infty(M)$ ,

$$((-\Delta)^{\frac{1}{2}} f, g) = (f, (-\Delta)^{\frac{1}{2}} g) = (f, (-\Delta)(-\Delta)^{-\frac{1}{2}} g) = (\nabla f, \nabla(-\Delta)^{-\frac{1}{2}} g).$$

Since the Riesz transform  $\nabla(-\Delta)^{-\frac{1}{2}}$  is  $L^r$  bounded for  $1 < r \leq 2$  (see [3, Theorem 1.1]),

$$|((-\Delta)^{\frac{1}{2}} f, g)| \leq \|\nabla f\|_p \|\nabla(-\Delta)^{-\frac{1}{2}} g\|_{p'} \leq C_{n,p} \|\nabla f\|_p \|g\|_{p'}.$$

Thus  $\|(-\Delta)^{\frac{1}{2}} f\|_p \leq C_{n,p} \|\nabla f\|_p$ .

Now we prove Lemmas 2.1-2.2. To this end, we give the following two lemmas which are needed in the proof of Lemma 2.1.

**Lemma 2.3.** *For any  $a > 0$ , let  $g_a(\theta) = \begin{cases} \theta^n, & 0 < \theta \leq 1, \\ \theta^a, & \theta > 1. \end{cases}$  Then there exists a constant  $C_{n,a} > 0$ , such that*

$$\sum_{k=-\infty}^h e^{-\frac{2^k}{5}} g_a(2^{\frac{k}{2}}) \leq C_{n,a} \int_0^{2^{h+1}} \frac{e^{-\frac{t}{5}} g_a(\sqrt{t})}{t} dt. \tag{2.7}$$

**Proof.** One can easily see that  $\lambda(t) = e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}})$  is a continuous and piecewise monotone function. Its maximum is obtained at

$$m = \begin{cases} 0, & a < \frac{2}{5}, \\ \log_2 \frac{5}{2} a, & a \geq \frac{2}{5}. \end{cases}$$

For  $h \geq m - 1$ , we have

$$\begin{aligned} \sum_{k=-\infty}^h e^{-\frac{2^k}{5}} g_a(2^{\frac{k}{2}}) &\leq \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt + e^{-\frac{2^m}{5}} g_a(2^{\frac{m}{2}}) \\ &\leq \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt + \frac{C'_a}{\int_{-\infty}^0 e^{-\frac{2^t}{5}} 2^{\frac{nt}{2}} dt} \int_{-\infty}^0 e^{-\frac{2^t}{5}} 2^{\frac{nt}{2}} dt \\ &\leq C_{n,a} \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt \leq C_{n,a} \int_0^{2^{h+1}} \frac{e^{-\frac{t}{5}} g_a(\sqrt{t})}{t} dt. \end{aligned} \tag{2.8}$$

While for  $h < m - 1$ ,

$$\sum_{k=-\infty}^h e^{-\frac{2^k}{5}} g_a(2^{\frac{k}{2}}) \leq \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt \leq \int_0^{2^{h+1}} \frac{e^{-\frac{t}{5}} g_a(\sqrt{t})}{t} dt. \tag{2.9}$$

Combining (2.8) and (2.9), we get the conclusion.

For simplicity, we denote  $g(\theta) = g_\nu(\theta)$ .

**Lemma 2.4.** *Let  $M$  be a complete Riemannian manifold satisfying condition (1.1). Then for any  $x \in M$ ,  $0 < s < 1$ , and  $\kappa > 0$ ,*

$$\int_{\rho(x,y) \leq \kappa} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) \leq C_{n,\nu} V(x, \sqrt{s}) \int_0^{\sqrt{2\kappa}} \frac{e^{-\frac{\rho^2}{5s}} g\left(\frac{\rho}{\sqrt{s}}\right)}{\rho} d\rho.$$

**Proof.** Since

$$\lim_{t \rightarrow 0} \frac{V(x, t)}{t^n} = \Omega_n > 0, \tag{2.10}$$

there exists an  $r_0 > 0$ , such that  $C_1 \leq \frac{V(x,s)}{s^n} \leq C_2$  for  $s \in (0, r_0)$ . Therefore for any  $r \in (0, r_0]$  and  $\theta \in (0, 1)$ ,

$$\frac{V(x, \theta r)}{V(x, r)} \leq C_n \theta^n. \tag{2.11}$$

On the other hand, for any  $r \in (r_0, 1]$ ,  $\theta \in (0, 1)$ ,

$$\frac{V(x, \theta r)}{V(x, r)} = \frac{V(x, \theta r)}{V(x, \theta r_0)} \cdot \frac{V(x, r_0)}{V(x, r)} \cdot \frac{V(x, \theta r_0)}{V(x, r_0)} \leq D_0 \left(\frac{r}{r_0}\right)^\nu \theta^n \leq C_{n,\nu} \theta^n. \tag{2.12}$$

For  $\theta \in (0, \infty)$ , we denote  $\chi(\theta) = \sup_{x \in M, 0 < r \leq 1} \frac{V(x, \theta r)}{V(x, r)}$ . Then it follows from (1.7), (2.11) and (2.12) that  $\chi(\theta) \leq C_{n,\nu} g(\theta)$ .

Let  $h$  be selected such that  $2^h \sqrt{s} \leq \kappa < 2^{h+1} \sqrt{s}$ . Thus

$$\begin{aligned} \int_{\rho(x,y) \leq \kappa} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) &\leq \sum_{k=-\infty}^h \int_{2^{\frac{k}{2}} \sqrt{s} < \rho(x,y) \leq 2^{\frac{k+1}{2}} \sqrt{s}} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) \\ &\leq \sum_{k=-\infty}^h e^{-\frac{2^k}{5}} (V(x, 2^{\frac{k+1}{2}} \sqrt{s}) - V(x, 2^{\frac{k}{2}} \sqrt{s})) \\ &\leq C_{n,\nu} V(x, \sqrt{s}) \sum_{k=-\infty}^h e^{-\frac{2^k}{5}} g(2^{\frac{k}{2}}). \end{aligned} \tag{2.13}$$

By Lemma 2.3,

$$\int_{\rho(x,y) \leq \kappa} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) \leq C_{n,\nu} V(x, \sqrt{s}) \int_0^{\sqrt{2\kappa}} \frac{e^{-\frac{\rho^2}{5s}} g\left(\frac{\rho}{\sqrt{s}}\right)}{\rho} d\rho.$$

And the lemma is proved.

**Proof of Lemma 2.1.** Let

$$k_{1,1}(x, y) = \begin{cases} k_1(x, y), & \rho(x, y) \leq \kappa, \\ 0, & \rho(x, y) > \kappa, \end{cases} \quad k_{1,\infty}(x, y) = \begin{cases} 0, & \rho(x, y) \leq \kappa, \\ k_1(x, y), & \rho(x, y) > \kappa, \end{cases}$$

where  $\kappa$  is to be determined.

Let

$$T_{1,1}f(x) = \int_M k_{1,1}(x, y)f(y)d\mu(y), \quad T_{1,\infty}f(x) = \int_M k_{1,\infty}(x, y)f(y)d\mu(y).$$

As long as we can show that  $T_1$  is of weak type  $(p, q)$ , it is also of  $(p, q)$  by Marcinkiewicz interpolation. To this aim, we need to prove

$$\mu\{x \in M : \|T_1f(x)\| > 2\lambda\} \leq C_{n,p,q,\nu} \frac{1}{\lambda^q}, \tag{2.14}$$

where  $\|f\|_p = 1$ .

Clearly

$$\mu\{x \in M : |T_1f(x)| > 2\lambda\} \leq \mu\{x \in M : |T_{1,1}f(x)| > \lambda\} + \mu\{x \in M : |T_{1,\infty}f(x)| > \lambda\}. \tag{2.15}$$

By interpolation,

$$\|T_{1,1}f(x)\|_p \leq \sup_{x \in M} \int_M k_{1,1}(x, y)d\mu(y). \tag{2.16}$$

By Lemma 2.4 and (1.6),

$$\begin{aligned} \int_M k_{1,1}(x, y)d\mu(y) &\leq \int_0^\infty te^{-t} \int_0^1 e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} \int_{\rho(x,y) \leq \kappa} \frac{e^{-\frac{\rho^2(x,y)}{5s}}}{V(x, \sqrt{s})} d\mu(y) ds dt \\ &\leq C_{n,\nu} \int_0^{+\infty} te^{-t} \int_0^1 e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} \int_0^{\sqrt{2\kappa}} \frac{e^{-\frac{\rho^2}{5s}} g\left(\frac{\rho}{\sqrt{s}}\right)}{\rho} d\rho ds dt \\ &\leq C_{n,\nu} \int_0^{\sqrt{2\kappa}} \frac{1}{\rho} \int_0^\infty te^{-t} \int_0^1 e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} e^{-\frac{\rho^2}{5s}} g\left(\frac{\rho}{\sqrt{s}}\right) ds dt d\rho \\ &\leq C_{n,\nu} \int_0^{\sqrt{2\kappa}} \frac{1}{\rho} \int_0^\infty e^{-t} \int_{t^2}^\infty e^{-\frac{s}{4}} s^{-\frac{1}{2}} e^{-\frac{s\rho^2}{5t^2}} g\left(\frac{\rho\sqrt{s}}{t}\right) ds dt d\rho \\ &\leq C_{n,\nu} \int_0^{\sqrt{2\kappa}} \frac{1}{\rho} \int_0^\infty e^{-\frac{s}{4}} s^{-\frac{1}{2}} \int_0^{\sqrt{s}} e^{-t} e^{-\frac{s\rho^2}{5t^2}} g\left(\frac{\rho\sqrt{s}}{t}\right) dt ds d\rho \\ &\leq C_{n,\nu} \int_0^{\sqrt{2\kappa}} \frac{1}{\rho} \int_0^\infty e^{-\frac{s}{4}} \int_0^\infty e^{-\frac{\sqrt{s}\rho}{t}} e^{-\frac{t^2}{5}} g(l) \frac{\rho}{l^2} dl ds d\rho. \end{aligned}$$

Since  $g(l) \sim l^n$  for  $l \rightarrow 0^+$ , and  $g(l) \sim l^\nu$  for  $l \rightarrow +\infty$ ,

$$\int_M k_{1,1}(x, y)d\mu(y) \leq C_{n,\nu}\kappa. \tag{2.17}$$

By (2.16) and (2.17),

$$\mu\{x \in M, |T_{1,1}f(x)| > \lambda\} \leq C_{n,\nu} \frac{\kappa^p}{\lambda^p}. \tag{2.18}$$

On the other hand,

$$|T_{1,\infty}f(x)| \leq \sup_{x \in M} \left( \int_M k_{1,\infty}^{p'}(x, y)d\mu(y) \right)^{\frac{1}{p'}}, \tag{2.19}$$

where  $\frac{1}{p'} = 1 - \frac{1}{p}$ .

By (2.10), there exists an  $r_0 > 0$ , such that for any  $s \in (0, r_0)$ ,  $V(x, \sqrt{s}) \geq c_n s^{\frac{n}{2}}$ . By (1.7) and  $\nu \geq n$ , for  $s \in (r_0, 1]$ ,

$$\frac{V(x, 1)}{V(x, \sqrt{s})} \leq D_0 \left(\frac{1}{\sqrt{s}}\right)^n \left(\frac{1}{\sqrt{s}}\right)^{\nu-n} \leq D_0 \left(\frac{1}{\sqrt{s}}\right)^n \left(\frac{1}{r_0}\right)^{\nu-n} \leq C_{n,\nu} s^{-\frac{n}{2}}.$$

Therefore

$$V(x, \sqrt{s}) \geq C_{n,\nu} \min(\delta, 1) s^{\frac{n}{2}}, \quad \forall 0 < s \leq 1. \tag{2.20}$$

It follows from (2.20) and [3, Lemma 2.1] that

$$\begin{aligned} \left(\int_M k_{1,\infty}^{p'}(x, y) d\mu(y)\right)^{\frac{1}{p'}} &\leq C \int_0^{+\infty} t e^{-t} \int_0^1 e^{-\frac{t^2}{4s} s^{-\frac{3}{2}}} \left(\int_{\rho(x,y) \geq \kappa} H^{p'}(x, y, s) d\mu(y)\right)^{\frac{1}{p'}} ds dt \\ &\leq C \int_0^{+\infty} t e^{-t} \int_0^1 \frac{e^{-\frac{t^2}{4s} s^{-\frac{3}{2}}}}{V(x, \sqrt{s})} \left(\int_{\rho(x,y) \geq \kappa} e^{-\frac{p' \rho^2(x,y)}{5s}} d\mu(y)\right)^{\frac{1}{p'}} ds dt \\ &\leq C \int_0^{+\infty} t e^{-t} \int_0^1 \frac{e^{-\frac{t^2}{4s} s^{-\frac{3}{2}}}}{V(x, \sqrt{s})} e^{-\frac{\kappa^2}{10s}} ds dt \\ &\leq C_{n,\nu} \frac{1}{\min\{\delta, 1\}} \int_0^{+\infty} t e^{-t} \int_0^1 e^{-\frac{t^2}{4s} s^{-\frac{3}{2}} - \frac{n}{2p}} e^{-\frac{\kappa^2}{10s}} ds dt. \end{aligned}$$

In a way Similar to the estimate of (2.17), we have

$$\left(\int_M k_{1,\infty}^{p'}(x, y) d\mu(y)\right)^{\frac{1}{p'}} \leq C_{p,n,\nu} \frac{1}{\min\{\delta, 1\}} \kappa^{1-\frac{n}{p}}.$$

By (2.19),

$$|T_{1,\infty} f(x)| \leq C_{p,n,\nu} \frac{1}{\min\{\delta, 1\}} \kappa^{1-\frac{n}{p}}. \tag{2.21}$$

Let  $\kappa > 0$  be selected such that

$$\lambda = C_{p,n,\nu} \frac{1}{\min\{\delta, 1\}} \kappa^{1-\frac{n}{p}} = C_{p,n} \frac{1}{\min\{\delta, 1\}} \kappa^{-\frac{n}{q}}.$$

Then

$$\kappa = C_{p,n,\nu,\delta} \lambda^{-\frac{q}{n}} \tag{2.22}$$

and (2.14) is obtained from (2.15), (2.18), (2.21) and (2.22). The lemma is proved.

**Proof of Lemma 2.2.** Write  $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{n} = \frac{n-1}{n}$ . By Hölder's inequality,

$$\begin{aligned} |T_2 f(x)| &\leq \int_M k_2(x, y) |f(y)| dy \\ &\leq \left\{ \int_M k_2^r(x, y) |f(y)|^p dy \right\}^{\frac{1}{q}} \cdot \left\{ \int_M k_2^r(x, y) dy \right\}^{1-\frac{1}{p}} \cdot \left\{ \int_M |f(y)|^p dy \right\}^{\frac{1}{n}}. \end{aligned}$$

Thus

$$\|T_2 f\|_q \leq \sup_{x \in M} \left\{ \int_M k_2^r(x, y) dy \right\}^{\frac{1}{r}} \|f\|_p. \quad (2.23)$$

By Mincowski's inequality, we have

$$\left( \int_M k_2^r(x, y) dy \right)^{\frac{1}{r}} \leq \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} \left( \int_M H^r(x, y, s) dy \right)^{\frac{1}{r}} ds dt.$$

Since  $V(x, \sqrt{s}) \geq V(x, 1) \geq \delta$  for  $s \geq 1$ ,

$$\left( \int_M H^r(x, y, s) d\mu(y) \right)^{\frac{1}{r}} \leq C_\delta.$$

Therefore

$$\left( \int_M k_2^r(x, y) d\mu(y) \right)^{\frac{1}{r}} \leq C_\delta \int_0^\infty t e^{-t} \int_1^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} ds dt \leq C_\delta.$$

Combining the above with (2.23), we get the lemma.

**Proof of Theorem 1.1.** By Theorem 2.1 and Theorem 2.2, we get the sufficient part of Theorem 1.1. Now let  $f(y) = \max\{t - \rho(x, y), 0\}$ . Then

$$\|f\|_q \geq \left(\frac{t}{2}\right) V\left(x, \frac{t}{2}\right)^{\frac{1}{q}}, \quad \|f\|_p \leq t V(x, t)^{\frac{1}{p}}, \quad \|\nabla f\|_p \leq V(x, t)^{\frac{1}{p}}.$$

Hence

$$V(x, t)^{\frac{1}{p} - \frac{1}{q}} \geq C_{n,p,q} \frac{t}{1+t}, \quad \forall t > 0. \quad (2.24)$$

Thus it follows from (2.10) and (2.24) that

$$t^{n(\frac{1}{p} - \frac{1}{q})} \geq C_{n,p,q} \frac{t}{1+t}, \quad \forall t \ll 1.$$

This means that  $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$ .

Choosing  $t = 1$  in (2.24), we get  $V(x, 1) \geq C_{n,p,q}$ ,  $\forall x \in M$ . Therefore  $\inf_{x \in M} V(x, 1) > 0$ .

And we get the necessary part of Theorem 1.1.

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