

ON RANDERS METRICS OF QUADRATIC RIEMANN CURVATURE

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In this paper, we study Randers metrics with quadratic Riemann curvature as in the Riemannian case. We find equations that characterize R-quadratic Randers metrics. In particular, we show that R-quadratic Randers metrics must have constant S-curvature.

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1. Introduction

Finsler metrics are Riemann metrics without quadratic restriction. For a Finsler metric $F = F(x, y)$, its locally minimizing curves are characterized by a system of differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the *spray coefficients*. If $F = \sqrt{g_{ij}(x)y^iy^j}$ is Riemannian, then $G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^jy^k$ are quadratic in $y \in T_xM$. This quadratic property is crucial in the regularity of the exponential map

$\exp_x : T_xM \rightarrow M$ at the origin of T_xM . Namely, \exp_x is C^∞ at the origin $0 \in T_xM$ at any point x if and only if the spray coefficients of F are quadratic in $y \in T_xM$ at any point x . There are non-Riemannian metrics whose spray coefficients still have this quadratic property. Finsler metrics with this property are called *Berwald metrics*. It is known that every Berwald metric has the same geodesics as a Riemannian metric [11]. Thus Berwald metrics can be identified with Riemannian metrics at geodesic level. The Riemann curvature is a family of linear maps $R_y = R^i_k \frac{\partial}{\partial x^k} \otimes dx^k : T_xM \rightarrow T_xM$, given by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

From the above formula, one can see that if F is a Berwald metric, then $R^i_k = R^i_k(x, y)$ are quadratic in $y \in T_xM$. Finsler metrics with such curvature property are called *R-quadratic metrics*. The notion of R-quadratic metrics is weaker than that of Berwald metrics. If a Finsler metric is of zero flag curvature, then it is a R-quadratic metric. If, in addition, it is not locally Minkowskian, then it is not Berwaldian. See Example 1.1 below.

Our goal is try to understand Finsler metrics whose Ricci/Riemann curvature has the quadratic property as Riemannian metrics. In general, it is a quite difficult problem. Thus we start with Randers metrics.

A Randers metric on a manifold is a Finsler metric in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1$. By the main theorem in [7], any R-quadratic Finsler metric on a closed manifold must be a Landsberg metric. On the other hand, a Randers metric $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel. In this case, it must be a Berwald metric. Thus we get the following

Theorem 1.1. *Let $F = \alpha + \beta$ be a positively complete Randers metric on a manifold. Then F is R-quadratic if and only if it is a Berwald metric.*

Let $\nabla\beta = b_{ij}y^i dx^j$ denote the covariant derivatives of β with respect to α .

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^i s_{ij}.$$

We denote $s^i_j := a^{ik} s_{kj}$. For a tensor with coefficients $T_{\dots ij}$ in local coordinates, we denote $T_{\dots i0} := T_{\dots ij}y^j$ and $T_{\dots 00} := T_{\dots ij}y^i y^j$, etc. We have the following

Theorem 1.2. *A Randers metric $F = \alpha + \beta$ on a manifold is R-quadratic if and only if*

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2), \tag{1.1}$$

$$2\alpha^2\Phi^i_{0k} - \alpha^2\Phi^i_{k0} - \Phi^i_{00}a_{jk}y^j = 0, \tag{1.2}$$

where $\Phi^i_{jk} := s^i_{j|k} - (2cs_j + c^2b_j + s_m s^m_j)\delta^i_k$ and c is constant.

If a Randers metric $F = \alpha + \beta$ satisfies (1.1) and (1.2), then the Riemann curvature of F is given by

$$R^i_k = \bar{R}^i_k + 3s^i_0 s_{k0} - (s^i_m s^m_k \alpha^2 - s^i_m s^m_0 a_{jk} y^j) + \Psi_0 \delta^i_k - \Psi_k y^i \tag{1.3}$$

where $\Psi_k := 3c^2 a_{jk} y^j - \beta c^2 b_k + s_0 s_k + 2s_{0|k} - s_{k|0} - 6cs_{k0}$. Clearly, R^i_k are quadratic in $y \in T_x M$.

In [1], Bacso–Matsumoto also study R-quadratic Randers metrics. But [1, Theorem 1] is inconsistent with Theorem 1.2 above. Bacso–Matsumoto’s arguments are based on Matsumoto’s paper on Randers metrics of constant curvature [5]. However, there is a gap in Matsumoto’s arguments and the conclusion in [5] is wrong. The mistake is corrected in [2]. However, the sufficient condition in [1, Theorem 1] is still true. See Corollary 1.3 below.

Corollary 1.3. *Let $F = \alpha + \beta$ be a Randers metric. If*

$$r_{ij} = 0, \quad s_j = 0, \quad s_{ij|k} = 0, \tag{1.4}$$

then F is R-quadratic.

It is difficult to solve the differential equations (1.1) and (1.2). Equation (1.1) is equivalent to that F has constant S-curvature, $S = (n + 1)cF$ [4]. We can express a Randers metric using navigation data as follows [8],

$$F = \frac{\sqrt{|y|_h^2 - (|W|_h^2 |y|_h^2 - \langle y, W \rangle_h^2)}}{1 - |W|_h^2} - \frac{\langle y, W \rangle_h}{1 - |W|_h^2}, \tag{1.5}$$

where $h = \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric and $W = W^i \frac{\partial}{\partial x^i}$ is a vector field; $\|\cdot\|_h$ and $\langle \cdot, \cdot \rangle_h$ denote the norm and inner product on $T_x M$ induced by h . According to [10], the S-curvature condition $S = (n + 1)cF$ is equivalent to that W is homothetic, i.e. $W_{i;j} + W_{j;i} = -4ch_{ij}$, where $W_i := h_{ij} W^j$ and $W_{i;j}$ denotes the covariant derivative with respect to h . By taking a Killing vector field $W = xQ + b$ on the Euclidean space with $h = |y|$, we obtain a Randers metric with vanishing flag curvature and vanishing S-curvature. In particular, it is R-quadratic.

Example 1.1. Consider the following Randers metric defined nearby the origin in R^n .

$$F := \frac{\sqrt{|y|^2 - (|xQ|^2 |y|^2 - \langle y, xQ \rangle^2)}}{1 - |xQ|^2} - \frac{\langle y, xQ \rangle}{1 - |xQ|^2},$$

where $Q = (q_j^i)$ is an anti-symmetric matrix. Then F satisfies (1.1) and (1.2) with $c = 0$. In fact, $R^i_k = 0$ but F is not a Berwald metric when $Q \neq 0$. This is a special example in the class of Randers metrics of zero flag curvature [3].

There are two weaker notions than R-quadratic metrics. The first one is the notion of Ricci-quadratic. We obtain a similar theorem that characterizes Ricci-quadratic Randers metrics. See Theorem 2.2 below. The second one is the notion of W-quadratic. We also obtain a similar theorem that characterizes W-quadratic Randers metrics. See Theorem 4.2 below.

2. Ricci-Quadratic Metrics

First let us recall a formula for the Riemann curvature of a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Let

$$q_{ij} := r_{im} s^m_j, \quad t_{ij} := s_{im} s^m_j, \quad t_j := b^i t_{ij} = s_m s^m_j.$$

Here and thereafter, we use a_{ij} to raise and lower the indices of tensors defined by b_i and $b_{i|j}$. Let $y_k := a_{jk} y^j$. Thus $y_0 = \alpha^2$. We have the following

Lemma 2.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M . Then the Riemann curvature of F is given by*

$$\begin{aligned} R^i_k &= \bar{R}^i_k + A y^i y_k + B y^i b_k + C \delta^i_k \\ &\quad + t^i_0 y_k - \alpha^2 t^i_k + 3 s^i_0 s_{k0} + \frac{\alpha^2}{F} t_k y^i - \frac{3}{F} s_0 s_{k0} y^i \\ &\quad + \frac{2\alpha}{F} q_{k0} y^i - \frac{\alpha}{F} q_{0k} y^i + 2\alpha s^i_{0|k} - \alpha s^i_{k|0} - \frac{1}{\alpha} s^i_{0|0} y_k \\ &\quad + \frac{\alpha}{F} s_{k0} y^i - \frac{2\alpha}{F} s_{0|k} y^i + \frac{1}{F} r_{00|k} y^i - \frac{1}{F} r_{k0|0} y^i, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} A &:= \left(\frac{2\alpha}{F^2} - \frac{1}{F}\right) t_0 - \left(\frac{2}{F^2} + \frac{1}{F\alpha}\right) q_{00} \\ &\quad + \frac{1}{F\alpha} s_{0|0} - \frac{3}{4F^3\alpha} (r_{00} - 2\alpha s_0)^2 + \frac{1}{2F^2\alpha} (r_{00|0} - 2\alpha s_{0|0}), \\ B &:= \frac{2\alpha^2}{F^2} t_0 - \frac{2\alpha}{F^2} q_{00} - \frac{3}{4F^3} (r_{00} - 2\alpha s_0)^2 + \frac{1}{2F^2} (r_{00|0} - 2\alpha s_{0|0}), \\ C &:= -\frac{2\alpha^2}{F} t_0 + \frac{2\alpha}{F} q_{00} + \frac{3}{4F^2} (r_{00} - 2\alpha s_0)^2 - \frac{1}{2F} (r_{00|0} - 2\alpha s_{0|0}). \end{aligned}$$

The formula (2.1) is not explicitly given in [2], but they have done all the computations.

By Lemma 2.1, we can easily get a formula for the Ricci curvature $\text{Ric} = R^m_m$.

$$\begin{aligned} \text{Ric} &= \frac{1}{4(\beta + \alpha)^2} \{ -4t^k_k \alpha^4 + 8(-t^k_k \beta + s^k_{0|k}) \alpha^3 - 8(n-1)t_0 \alpha^3 \\ &\quad + [4(n-1)(s_{0|0} + 2q_{00} - 2t_0 \beta + 3s_0^2) \\ &\quad + 4(-t^k_k \beta^2 + \bar{\text{Ric}} + 4s^k_{0|k} \beta - 2t_{00})] \alpha^2 \\ &\quad + [(n-1)(-2r_{00|0} - 12s_0 r_{00} + 4\beta s_{0|0} + 8q_{00} \beta) \\ &\quad + 8s^k_{0|k} \beta^2 + 8\bar{\text{Ric}} - 16t_{00} \beta] \alpha \\ &\quad - 2(n-1)r_{00|0} \beta + 3(n-1)r_{00}^2 + 4\bar{\text{Ric}} \beta^2 - 8t_{00} \beta^2 \}, \end{aligned} \tag{2.2}$$

where $\bar{\text{Ric}} := \bar{R}^m_m$ denotes the Ricci curvature of α .

Let

$$A_k := 2cs_k + c^2b_k + t_k + \frac{1}{2}c_k,$$

$$\Psi_k := 3c^2y_k - c^2\beta b_k + 2\beta c_k - c_0b_k + s_0s_k + 2s_{0|k} - s_{k|0} - 6cs_{k0}.$$

Theorem 2.2. *Let $F = \alpha + \beta$ be a Randers metric on an n -manifold. Then it is Ricci-quadratic if and only if*

$$r_{00} + 2s_0\beta = 2c(\alpha^2 - \beta^2), \tag{2.3}$$

$$s^k_{0|k} = (n - 1)A_0, \tag{2.4}$$

where $c = c(x)$ is a scalar function. In this case,

$$\text{Ric} = \overline{\text{Ric}} - 2t_{00} - t^k_k\alpha^2 + (n - 1)\Psi_0. \tag{2.5}$$

Proof. Assume that F is Ricci-quadratic, that is, Ric is quadratic in $y \in T_xM$. Then (2.2) is equivalent to the following two equations.

$$2[s^k_{0|k} - t^k_k\beta - (n - 1)t_0]\alpha^2 + (n - 1) \left[-\frac{1}{2}r_{00|0} + 2q_{00}\beta - 3s_0r_{00} + \beta s_{0|0} \right] + 2s^k_{0|k}\beta^2 - 2\text{Ric}\beta + 2\overline{\text{Ric}}\beta - 4t_{00}\beta = 0, \tag{2.6}$$

$$-t^k_k\alpha^4 + [(n - 1)(s_{0|0} + 2q_{00} - 2t_0\beta + 3s_0^2) - 2t_{00} + \overline{\text{Ric}} + 4s^k_{0|k}\beta - \text{Ric} - t^k_k\beta^2]\alpha^2 + \frac{1}{4}(n - 1)(-2r_{00|0}\beta + 3r_{00}^2) + \overline{\text{Ric}}\beta^2 - 2t_{00}\beta^2 - \text{Ric}\beta^2 = 0. \tag{2.7}$$

(2.6) $\times \beta -$ (2.7) yields

$$-\frac{3(n - 1)}{4}(r_{00} + 2s_0\beta)^2 - (\alpha^2 - \beta^2)\{-t^k_k\alpha^2 + 2s^k_{0|k}\beta + (n - 1)(2q_{00} + s_{0|0} + 3s_0^2) - 2t_{00} + \overline{\text{Ric}} - \text{Ric}\} = 0. \tag{2.8}$$

Thus there exists some scalar function $c = c(x)$ such that (2.3) holds. By (2.3), we have

$$r_{00|0} = -2\beta s_{0|0} + 4s_0^2\beta + 8cs_0\beta^2 + (2c_0 - 4cs_0 - 8\beta c^2)(\alpha^2 - \beta^2), \tag{2.9}$$

$$q_{00} = -s_0^2 - t_0\beta - 2cs_0\beta. \tag{2.10}$$

Then the Ricci curvature becomes

$$\text{Ric} = \overline{\text{Ric}} - 2t_{00} - t^k_k\alpha^2 + (n - 1)(3c^2\alpha^2 + s_0^2 - c^2\beta^2 + \beta c_0 + s_{0|0}) + 2\{s^k_{0|k} - (n - 1)A_0\}\alpha. \tag{2.11}$$

Since Ric is quadratic in y , we see that the coefficient of α must be zero, that is, (2.4) holds. Then (2.11) is reduced to (2.5). □

3. R-Quadratic Metrics

Now we can simplify the formula (2.1) for R^i_k under the condition (2.3).

$$R^i_k = \bar{R}^i_k + 3s^i_0 s_{k0} - (t^i_k \alpha^2 - t^i_0 y_k) + \Psi_0 \delta^i_k - \Psi_k y^i + \alpha^{-1} \left\{ (2\alpha^2 \Phi^i_{0k} - \alpha^2 \Phi^i_{k0} - \Phi^i_{00} y_k) + \frac{3}{2} (\alpha^2 c_k - c_0 y_k) y^i \right\}, \quad (3.1)$$

where $\Phi^i_{jk} := s^i_{j|k} - A_j \delta^i_k$.

By (3.1), we obtain the following

Lemma 3.1. *A Randers metric $F = \alpha + \beta$ is R-quadratic if and only if*

$$r_{00} + 2s_0 \beta = 2c(\alpha^2 - \beta^2), \quad (3.2)$$

$$(2\alpha^2 \Phi^i_{0k} - \alpha^2 \Phi^i_{k0} - \Phi^i_{00} y_k) + \frac{3}{2} (\alpha^2 c_k - c_0 y_k) y^i = 0, \quad (3.3)$$

where $c = c(x)$ is a scalar function. In this case

$$R^i_k = \bar{R}^i_k + 3s^i_0 s_{k0} - (t^i_k \alpha^2 - t^i_0 y_k) + \Psi_0 \delta^i_k - \Psi_k y^i. \quad (3.4)$$

We shall show that $c = c(x)$ in (3.2) is actually constant.

Lemma 3.2. *If $F = \alpha + \beta$ is R-quadratic, then $c = \text{constant}$.*

Proof. Contracting (3.3) with y_i yields

$$-(A_0 y_k - \alpha^2 A_k) - s_{0k|0} + \frac{3}{2} (\alpha^2 c_k - c_0 y_k) = 0. \quad (3.5)$$

We get

$$s^i_{0|0} = (A_0 y^i - \alpha^2 A^i) - \frac{3}{2} (\alpha^2 c^i - c_0 y^i). \quad (3.6)$$

Plugging it into (3.3) yields

$$\alpha^2 \{ (2s^i_{0|k} - s^i_{k|0}) - (2A_0 \delta^i_k - A_k y^i - A^i y_k) + \frac{3}{2} (c_k y^i + c^i y_k) \} = 3c_0 y_k y^i.$$

Clearly, $c_0 = 0$. Thus $c = \text{constant}$. □

4. W-Quadratic Metrics

Let $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ denote the Riemann curvature of a Finsler metric F . Let

$$A^i_k := R^i_k - R \delta^i_k, \quad R := \frac{R^m_m}{n-1}.$$

Then the (projective) Weyl curvature $W_y = W^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$W^i_k := A^i_k - \frac{1}{n+1} \frac{\partial A^m_k}{\partial y^m} y^i.$$

A Finsler metric with W^i_k quadratic in y is said to be *W-quadratic*. Note that if R^i_k are R-quadratic, then W^i_k are quadratic in y . Namely, every R-quadratic Finsler metric must be W-quadratic. Thus it is a natural problem to study W-quadratic Finsler metrics.

First we recall a formula for the Weyl curvature.

Lemma 4.1 ([9]). *For a Randers metric $F = \alpha + \beta$,*

$$\begin{aligned} W^i_k &= \bar{W}^i_k + 3s^i_0 s_{k0} + t^i_0 y_k - \alpha^2 t^i_k \\ &+ \frac{1}{n-1} \{ (2t_{00} + \alpha^2 t^m_m) \delta^i_k - (2t_{k0} + t^m_m y_k) y^i \} \\ &+ \alpha^{-1} \left\{ (2\alpha^2 s^i_{0|k} - \alpha^2 s^i_{k|0} - s^i_{0|0} y_k) \right. \\ &\left. - \frac{1}{n-1} (2\alpha^2 s^m_{0|m} \delta^i_k - s^m_{0|m} y_k y^i - \alpha^2 s^m_{k|m} y^i) \right\}, \end{aligned}$$

where \bar{W}^i_k denote the Weyl curvature of α .

By Lemma 4.1, we immediately obtain

Theorem 4.2. *Let $F = \alpha + \beta$ be a Randers metric on an n -manifold. It is W-quadratic if and only if*

$$2\alpha^2 s^i_{0|k} - \alpha^2 s^i_{k|0} - s^i_{0|0} y_k = \frac{1}{n-1} \{ 2\alpha^2 s^m_{0|m} \delta^i_k - s^m_{0|m} y_k y^i - \alpha^2 s^m_{k|m} y^i \}. \tag{4.1}$$

We have the following corollaries.

Corollary 4.3. *Let $F = \alpha + \beta$ be a Randers metric on an n -manifold. If*

$$s_{ij|k} = \frac{1}{n-1} \{ a_{ik} s^m_{j|m} - a_{jk} s^m_{i|m} \}, \tag{4.2}$$

then it is W-quadratic.

Proof. It follows from (4.2) that

$$\begin{aligned} s^i_{0|k} &= \frac{1}{n-1} \{ \delta^i_k s^m_{0|m} - y_k s^m_{i|m} \}, \\ s^i_{k|0} &= \frac{1}{n-1} \{ y^i s^m_{k|m} - y_k s^m_{i|m} \}, \\ s^i_{0|0} &= \frac{1}{n-1} \{ y^i s^m_{0|m} - \alpha^2 s^m_{i|m} \}. \end{aligned}$$

Then (4.1) holds. □

Corollary 4.4. *If (3.3) holds, then F is W-quadratic.*

Proof. It follows from (3.3) that

$$A_k = \frac{1}{n-1} s_{k|m}^m.$$

Then (3.3) can be written as (4.1). □

Finally, we make the following

Conjecture 4.5. Let $F = \alpha + \beta$ be a Randers metric on a closed manifold. If it is W -quadratic, then β is parallel.

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